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Equations admitting $O(2, 1)\otimes\mathbb{R}(t, t^{-1})$ as a prolongation algebra

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Abstract. We derive several equations possessing $O(2, 1)\otimes\mathbb{R}(t, t^{-1})$ as a prolongation algebra. An example of a Bäcklund transformation is given.

1. Introduction

In a sequel to earlier papers [1-3] where we found the inifinite-dimensional prolongation algebra for some specific non-linear differential equations we shall here continue our studies of prolongation structures by going in the opposite direction. There the question was: given a non-linear differential equation, does it admit an infinite-dimensional algebra and how can that be found? Here now we turn the question around and ask: given an infinite-dimensional Lie algebra, is it the prolongation algebra of some non-linear differential equation and how can that be found?

As can be inferred from [2] there are many equations to be derived from a particular algebra and in this paper we shall take $O(2, 1)\otimes\mathbb{R}(t, t^{-1})$ as an example. In [2] this algebra was shown to be the prolongation algebra to the Korteweg-de Vries and non-linear Schrödinger equations. In § 3 we shall demonstrate that many more equations, some of them well known, some of them to our knowledge new, can be derived from it.

In [1, 2] we have also found other prolongation algebras and it is not obvious how they are connected with that of the present paper. In § 2 we shall therefore show that the algebras found there are isomorphic to subalgebras of $O(3)\otimes\mathbb{R}(t, t^{-1})$ or to $O(2, 1)\otimes\mathbb{R}(t, t^{-1})$ itself.

Prolongation algebras are useful for finding Bäcklund transformations; a particular instance will be discussed in § 4. Furthermore they are useful for finding relations between apparently different equations such as the Miura transformation (not to be discussed here; examples have been given in [2]) or transformations like that connecting the Harry-Dym equation with the modified Korteweg-de Vries equation. The examples to be discussed in § 3 are chosen to show the various possibilities within the prolongation structure approach.

Neither in § 3 nor in § 4 shall we display all the details of the calculations. For some examples we describe the essential considerations from which the reader can fill in the missing steps; in other instances we shall just state the results.

Other methods have been used to derive non-linear equations from infinite-dimensional algebras [4, 5]. In particular Ablowitz *et al* [6, 7] have used the loop

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algebra $A_1^{(1)}$ of which our algebra is a real form. A comparison between their approach and ours will be given in § 5.

Before we go into details, let us outline the general formalism. All our considerations will be purely local and we will deal only with equations in two dimensions. Consider a set of non-linear first-order differential equations for n functions, $u(x, t)$. Any higher-order equation can be written as such a set. Now introduce 2-forms F on an $(n+2)$-dimensional manifold with coordinates (u, x, t) such that restriction to the submanifold labelled by x, t reproduces the original equations. Or, phrased equivalently, let $\phi: (x, t) \to (u, x, t)$. Then the F are to be chosen such that

$$\phi^* F = 0 = \text{given differential equation.} \tag{1.1}$$

In general the differential equations will have to satisfy integrability conditions. In the language of our forms F those conditions are

$$\mathrm{d}F = 0 \bmod F. \tag{1.2}$$

A set of such forms is called a closed ideal.

Now we enlarge the dimension of the manifold on which our forms live to an as yet unspecified number by introducing pseudopotentials or prolongation variables. To this end consider the 1-forms

$$\omega = -\mathrm{d}y + A(u, y, x, t)\,\mathrm{d}x + B(u, y, x, t)\,\mathrm{d}t \tag{1.3}$$

with an unspecified number of y. Let now ϕ be a map $\phi: (x, t) \to (y, u, x, t)$; the ω are to be chosen such that

$$\begin{aligned} \phi^* F &= 0 \\ \phi^* \omega &= 0 \end{aligned} \tag{1.4}$$

give integrable differential equations. The condition for this is

$$\mathrm{d}\omega = 0 \bmod (F, \omega). \tag{1.5}$$

Inserting ω into this equation we obtain terms

$$A_y\,\mathrm{d}y\,\mathrm{d}t + B_y\,\mathrm{d}y\,\mathrm{d}x = (B_y A - A_y B)\,\mathrm{d}x\,\mathrm{d}t \bmod \omega.$$

(We omit the wedge which normally appears between differential forms.) Other terms are of the form $A_u\,\mathrm{d}u\,\mathrm{d}x$ and some of them can be converted to $A_u f(u)\,\mathrm{d}x\,\mathrm{d}t \bmod F$. Finally we are left with equations of the type (a subscript denotes differentiation with respect to the indicated variable)

$$\begin{aligned} [AB] &= A_u f(u, x, t) + B_u g(u, x, t) \\ 0 &= A_u f'(u, x, t) + B_u g'(u, x, t) \end{aligned} \tag{1.6}$$

where the square bracket denotes the usual vector commutators $A^\alpha_{y^\beta} B^\beta - B^\alpha_{y^\beta} A^\beta$ with respect to the prolongation variables. Suppose that the (u, x, t) dependence of A and B can be determined from the above equations and that they can be written as (for some natural number n and m)

$$\begin{aligned} A &= \sum_{i=1}^{n} A^i(u, x, t) X_i(y) \\ B &= \sum_{i=n+1}^{m} B^i(u, x, t) X_i(y). \end{aligned} \tag{1.7}$$

Inserting (1.7) into (1.6) yields some, but in general not all, commutators between the X_i. As the Jacobi identities have to be satisfied one can use them to determine some, possibly all, of the unknown commutators. It may even turn out that some of the X_i have to vanish. One then introduces new generators equal to the still unknown commutators and repeats the process of going through the Jacobi identities. If this process appears to be open-ended, one can try to deduce the structure of the emerging infinite-dimensional algebra. For practical purposes an infinite-dimensional algebra is not manageable; one can then look for homomorphisms into a finite-dimensional one.

A Bäcklund transformation—more precisely an auto-Bäcklund transformation, i.e. one which maps solutions of an equation into solutions of the same equation—is in the present context a map $\phi:(u, y, x, t)\to(\tilde{u}, x, t)$ such that

$$\phi^*\tilde{F}=0 \bmod(F, \omega). \tag{1.8}$$

Let us now play the game the other way. We start with an infinite-dimensional Lie algebra

$$[X_i, X_k]=C^l_{ik}X_l$$

of vector fields on an infinite-dimensional manifold with coordinates y. Then introduce a basis of 1-forms dual to the basis vector fields X_i and define

$$\omega=-\mathrm{d}y+X_i\xi^i.$$

The 2-forms

$$\mathrm{d}\omega \bmod \omega = X_i(\mathrm{d}\xi^i-\tfrac{1}{2}C^i_{ke}\xi^k\xi^l)$$

define the Cartan–Maurer structure forms

$$F=\mathrm{d}\xi^i-\tfrac{1}{2}C^i_{ke}\xi^k\xi^l \tag{1.9}$$

of the algebra. How can we obtain differential equations out of this system? To this end, set all but a finite number of the ξ to zero, i.e. $\xi^i=0$, $i\notin N$ (a finite set of integers). This splits the set (1.9) naturally into two sets, the Ω, Σ system [4]:

$$\Omega=\mathrm{d}\xi^i-\tfrac{1}{2}C^i_{ke}\xi^k\xi^l \qquad i\in N \tag{1.10a}$$

$$\Sigma=C^i_{ke}\xi^k\xi^l \qquad i\notin N. \tag{1.10b}$$

The fact that the structure constants satisfy the Jacobi identities guarantees that

$$\mathrm{d}\Omega=0 \bmod(\Omega, \Sigma) \qquad \mathrm{d}\Sigma=0 \bmod(\Omega, \Sigma).$$

Ω, Σ are thus a closed ideal and the equations

$$\Omega=0 \tag{1.11a}$$

$$\Sigma=0 \tag{1.11b}$$

are integrable. The dimension of the integral manifold of the Ω, Σ system is Cartan's genus g and is the number of independent 1-forms admitted by $\Sigma=0$. We can thus write the ξ^i as linear combinations of those independent 1-forms.

For instance, let one of the equations (1.11*b*) be $\xi^1\xi^2=0$, then we solve it by writing $\xi^1=a\xi^2$. If in addition another of the equations (1.11*b*) is

$$\xi^1\xi^3-\xi^2\xi^4=0\Rightarrow\xi^2(a\xi^3-\xi^4)=0$$

then it can be solved by $\xi^4 = a\xi^3 + b\xi^2$. In general, the solution of (1.11b) can be written as

$$\xi^i = A^i_k \eta^k \qquad i \in N;\ 1 \leq k \leq g \tag{1.12}$$

where η^k are the independent 1-forms and A^i_k are functions of the coordinates of the integral manifold.

How can we find the coordinates? To this end, we substitute (1.12) into the Ω equations (1.11a) and look for exact 1-forms. For example, it may turn out that one such equation is $\mathrm{d}\eta^1 = 0$ and hence we set $\eta^1 = \mathrm{d}x^1$.

Suppose that we have found g independent exact 1-forms. Taking them as coordinate differentials, we evaluate (1.11a) and thus obtain a set of non-linear first-order differential equations which can then be combined to higher-order equations.

On the other hand, there could be more than g exact 1-forms, with, of course, only g of them independent. It is now our choice which subset of them to take as coordinates and thereby derive various equations from the same Ω, Σ system. Those equations are related by an interchange of coordinates (those 1-forms which we took to be the independent ones) and potentials (the other ones).

The Ω, Σ system could be such that we can only find less than g exact 1-forms. In this case we reduce the genus by the following method [4]. Take a linear combination with constant coefficients $\zeta = c_i \xi^i$ such that

$$\zeta\, \mathrm{d}\zeta = 0 \bmod(\Omega, \Sigma). \tag{1.13}$$

The equation $\zeta = 0$ gives one ξ in terms of the others; the genus and the number of Ω equations is reduced by one. This is equivalent to a change of base in the original algebra and to putting one more ξ to zero.

This reduction can, of course, also be effected if we have g exact 1-forms but want to lower the genus nevertheless. In this case we turn an independent 1-form (a coordinate) into a dependent one (a potential). The resulting equations will be quite different. The Miura transformation between the Korteweg–de Vries and the modified Korteweg–de Vries equations results from just such a reduction. Those possibilities will be exemplified in § 3.

Is the choice of N (i.e. which forms to keep and which to set to zero) restricted in any way? There is to our knowledge no *a priori* criterion; the choice is arbitrary and can only be justified *a posteriori*.

Another question is whether it is possible to reconstruct the infinite algebra from the Ω, Σ system. This system translates back into an incomplete Lie algebra with generators which are dual to those ξ which were kept non-zero. Treating this incomplete algebra in the usual manner which has been mentioned above finally gives that subalgebra generated by the basic generators. This algebra may be identical with or isomorphic to the whole algebra.

Closely related is a question concerning the uniqueness of the Ω, Σ system or, equivalently, whether the same equation can be derived from different systems. There is no obvious answer; the only thing which can be said is that if two sets of ξ are dual to generators belonging to certain subalgebras, if those subalgebras are isomorphic and if the basic generators are mapped into each other under the isomorphism, then the two Ω, Σ systems yield the same equation.

As a final remark, observe that $\mathrm{d}\omega = 0 \bmod(\Omega, \Sigma, \omega)$. Thus

$$\omega = 0 \tag{1.14}$$

gives the prolongation variables which we can then use to find Bäcklund transformations.

2. A remark on the algebras

In [1, 2] we have encountered prolongation algebras which generically look like

$$[A_i B_k] = C_{i+k+c} \qquad [B_i C_K] = A_{i+k+a} \qquad [C_i A_k] = B_{i+k+b} \tag{2.1}$$

where the A, B and C commute among themselves and a, b and c are given integers. Now consider the transformation

$$\hat{A}_i = A_i - [(b+c)/2] \qquad \hat{B}_i = B_i - [(a+c)/2] \qquad \hat{C}_i = C_i - [(a+b)/2].$$

There are two cases to distinguish. Either a, b and c are all even/odd or two of them are even/odd and the other one odd/even. In the first case we obtain an algebra like (2.1) with the new $\hat{a}$, $\hat{b}$, $\hat{c}$ all vanishing. The second case gives an algebra with one non-vanishing shift index equal to 1 and we take without loss of generality

$$[\hat{A}_i \hat{B}_k] = \hat{C}_{i+k} \qquad [\hat{B}_i \hat{C}_k] = \hat{A}_{i+k+1} \qquad [\hat{C}_i \hat{A}_k] = \varepsilon \hat{B}_{1+k}. \tag{2.2}$$

Here we have introduced the relative sign $\varepsilon = \pm 1$ because that becomes important for the next argument.

The algebra $O(3) \otimes \mathbb{R}(t, t^{-1})$, respectively $O(2, 1) \otimes \mathbb{R}(t, t^{-1})$, is given by

$$[X_i Y_k] = Z_{i+k} \qquad [Y_i Z_k] = X_{i+k} \qquad [Z_i X_k] = \varepsilon Y_{i+k} \tag{2.3}$$

for $\varepsilon = \pm 1$ where again the X, Y and Z commute among themselves. Both are real forms of the loop algebra of sl(2, C) or $A_1^{(1)}$. The relations

$$\hat{A}_i = X_{2i} \qquad \hat{B}_i = Y_{2i+1} \qquad \hat{C}_i = Z_{2i+1} \tag{2.4}$$

show that (2.2) is a subalgebra of (2.3) irrespective of ε.

On the other hand, for $\varepsilon = -1$ the relations

$$\begin{aligned} \hat{A}_i &= X_i \\ \hat{B}_i &= \tfrac{1}{2}(Y_i + Z_i + Y_{i+1} - Z_{i+1}) \\ \hat{C}_i &= \tfrac{1}{2}(Y_i + Z_i - Y_{i+1} + Z_{i+1}) \end{aligned} \tag{2.5}$$

give an isomorphic map between (2.2) and (2.3).

We have thus shown that any algebra (2.2) is either isomorphic to the subalgebra of (2.3) generated by even X and odd Y and Z or isomorphic to (2.3) itself.

There are homomorphic maps from the infinite-dimensional algebra (2.3) into $3n$-dimensional ones. They are generated by the recursion relations

$$G_{i+n} = \sum_{k=0}^{n-1} a_k G_{i+k} \tag{2.6}$$

where G stands for X, Y or Z. We pick any n consecutive G and determine those above and below them by (2.6). For later use we give explicitly the cases

$$n = 1 \qquad \begin{cases} G_{i+1} = \alpha G_i \\ G_i = \alpha^i G_0 \end{cases} \tag{2.7}$$

$$n = 2 \qquad G_{i+2} = 2\mu G_{i+1} + \nu G_i. \tag{2.8}$$

If $\mu^2 + \nu = \lambda^2 > 0$ the resulting six-dimensional algebra is $O(2, 1) \otimes O(2, 1)$.

3. Equations derivable from $O(2,1)\otimes\mathbb{R}(t, t^{-1})$

We take the algebra under consideration as

$$[X_i Y_k] = Y_{i+k} \qquad [X_i Z_k] = -Z_{i+k} \qquad [Y_i Z_k] = X_{i+k}$$

and denote the forms dual to the generators by

$$\begin{array}{lll} \xi^1 \text{ dual to } X_{-1} & \xi^2 \text{ dual to } Y_{-1} & \xi^3 \text{ dual to } Z_{-1} \\ \xi^4 \text{ dual to } X_0 & \xi^5 \text{ dual to } Y_0 & \xi^6 \text{ dual to } Z_0 \\ \xi^7 \text{ dual to } X_1 & \xi^8 \text{ dual to } Y_1 & \xi^9 \text{ dual to } Z_1 \\ \xi^{10} \text{ dual to } X_2 & \xi^{11} \text{ dual to } Y_2 & \xi^{12} \text{ dual to } Z_2 \\ \xi^{13} \text{ dual to } X_3. & & \end{array} \tag{3.1}$$

All forms dual to the other generators are supposed to vanish. As mentioned above, this choice is arbitrary and motivated only by the fact that it contains all the systems which we should like to discuss below.

The Ω, Σ system (1.11a) then becomes

$$\begin{array}{lll} d\xi^1 = \xi^2\xi^6 - \xi^3\xi^5 & d\xi^2 = \xi^1\xi^5 - \xi^2\xi^4 & d\xi^3 = -\xi^1\xi^6 + \xi^3\xi^4 \\ d\xi^4 = \xi^2\xi^9 - \xi^3\xi^8 + \xi^5\xi^6 & d\xi^5 = \xi^1\xi^8 - \xi^2\xi^7 + \xi^4\xi^5 & d\xi^6 = -\xi^1\xi^9 + \xi^3\xi^7 - \xi^4\xi^6 \\ d\xi^7 = \xi^2\xi^{12} - \xi^3\xi^{11} + \xi^5\xi^9 - \xi^6\xi^8 & d\xi^8 = \xi^1\xi^{11} - \xi^2\xi^{10} + \xi^4\xi^8 - \xi^5\xi^7 & \\ d\xi^9 = -\xi^1\xi^{12} + \xi^3\xi^{10} - \xi^4\xi^9 + \xi^6\xi^7 & d\xi^{10} = \xi^5\xi^{12} - \xi^6\xi^{11} + \xi^8\xi^9 & \\ d\xi^{11} = -\xi^2\xi^{13} + \xi^4\xi^{11} - \xi^5\xi^{10} + \xi^7\xi^8 & d\xi^{12} = \xi^3\xi^{13} - \xi^4\xi^{12} + \xi^6\xi^{10} - \xi^7\xi^9 & \\ d\xi^{13} = \xi^8\xi^{12} - \xi^9\xi^{11} & & \end{array} \tag{3.2a}$$

and

$$\begin{array}{lll} \xi^1\xi^2 = 0 & \xi^1\xi^3 = 0 & \xi^2\xi^3 = 0 \\ -\xi^5\xi^{13} + \xi^7\xi^{11} - \xi^8\xi^{10} = 0 & \xi^6\xi^{13} - \xi^7\xi^{12} + \xi^9\xi^{10} = 0 & \\ -\xi^8\xi^{13} + \xi^{10}\xi^{11} = 0 & \xi^9\xi^{13} - \xi^{10}\xi^{12} = 0 & \\ \xi^{11}\xi^{12} = 0 & \xi^{11}\xi^{13} = 0 & \xi^{12}\xi^{13} = 0. \end{array} \tag{3.2b}$$

In what follows we shall not use the full set of equations (3.2) but rather restrict ourselves to certain subsets of them; we shall indicate the non-vanishing ξ without spelling out the corresponding Ω, Σ system again. The examples are chosen such that they demonstrate various possibilities within the present formalism. For instance, the following cases (A)-(D) have been dealt with in [1, 2].

(A) $\qquad \xi^4 \sim \xi^{10} \neq 0$

gives the real version of the non-linear Schrödinger equation

$$a_t = a_{xx} - a^2 b \qquad b_t = -b_{xx} + ab^2.$$

(B) $\qquad \xi^4 \sim \xi^9 \qquad \xi^{11} = \xi^9 \neq 0$

gives the Korteweg-de Vries equation in the form

$$u_t = u_{xxx} + \tfrac{3}{2}u_x^2.$$

Furthermore, considering the remarks in the previous section, it follows from [2] that

(C) $\xi^6 \sim \xi^9 \qquad \xi^{11} \neq 0$

gives the modified Korteweg-de Vries equation with a different sign of the non-linear term

$$u_t = u_{xxx} - \tfrac{1}{2}u_x^3.$$

Similarly it can be concluded from [1] that

(D) $\xi^3 \sim \xi^6 \qquad \xi^8 \neq 0 \qquad \xi^3 = \xi^5$

yields the sinh-Gordon equation

$$\phi_{xt} = \mathrm{e}^{\phi} - \mathrm{e}^{-\phi}.$$

Note that the non-linear Schrödinger equation was obtained by retaining the forms dual to G_0, G_1, X_2. Which equation do we obtain if we keep the forms dual to G_0, G_1, G_2, X_3? The resulting system has genus four and the easiest way to reduce the genus is by setting $\xi^{10} = 0$. Hence we have

(E) $\xi^4 \sim \xi^9 \qquad \xi^{11} \sim \xi^{13} \neq 0.$

It follows immediately from (3.2b) that ξ^8, ξ^9, ξ^{11}, ξ^{12} and ξ^{13} have to be proportional and furthermore that ξ^5 and ξ^6 are linear combinations of ξ^7 and ξ^{13}. The solution of (3.2b) is given by

$$\xi^5 = a\xi^7 + g\xi^{13} \qquad \xi^6 = b\xi^7 + h\xi^{13} \qquad \xi^8 = e\xi^{13}$$

$$\xi^9 = f\xi^{13} \qquad \xi^{11} = a\xi^{13} \qquad \xi^{12} = b\xi^{13}.$$

The remaining independent forms are ξ^4, ξ^7 and ξ^{13}. By inserting the above expressions into (3.2a) it follows that

$$\mathrm{d}\xi^{13} = 0 \rightarrow \xi^{13} = \mathrm{d}t.$$

Now one eliminates the $\xi^4\xi^{13}$ terms between $\mathrm{d}\xi^{11}$ and $\mathrm{d}\xi^{12}$ and uses the result to show that

$$\mathrm{d}(\xi^7 + ab\xi^{13}) = 0 \qquad \xi^7 = \mathrm{d}x - ab\,\mathrm{d}t.$$

Similarly, by eliminating the various terms on the right-hand side of the $\mathrm{d}\xi^4$ equation one shows that

$$\mathrm{d}[\xi^4 + (af + be)\xi^{13}] = 0.$$

Hence all forms can be written as

$$\begin{aligned}
&\xi^4 = \mathrm{d}z - (af + be)\,\mathrm{d}t && \xi^8 = e\,\mathrm{d}t \\
&\xi^5 = a\,\mathrm{d}x + (g - a^2 b)\,\mathrm{d}t && \xi^9 = f\,\mathrm{d}t \\
&\xi^6 = b\,\mathrm{d}x + (h - ab^2)\,\mathrm{d}t && \xi^{11} = a\,\mathrm{d}t \\
&\xi^7 = \mathrm{d}x - ab\,\mathrm{d}t && \xi^{12} = b\,\mathrm{d}t \\
& && \xi^{13} = \mathrm{d}t.
\end{aligned} \tag{3.3}$$

Using x, z, t as coordinates (i.e. taking $a(x, z, t)$, etc) we get from (3.2a) by collecting terms in $\mathrm{d}x\,\mathrm{d}t$, etc, the following equations:

$$a_z = a \qquad b_z = -b \qquad e = a_x \qquad f = -b_x \qquad e_z = e$$

$$f_z = -f \qquad g = e_x \qquad h = -f_x \qquad g_z = g \qquad h_z = -h$$

$$-a_t + g_x - (a^2 b)_x = a(af + eb) \qquad -b_t + h_x - (ab^2)_x = -b(af + eb).$$

Putting everything together yields

$$a_t = a_{xxx} - 3aba_x \qquad b_t = b_{xxx} - 3abbx. \tag{3.4}$$

The z dependence just reflects the fact that (3.4) is invariant under a scaling of a by a constant and of b by its inverse.

From the point of view of the remaining 1-forms the system (3.4) is a generalisation of the non-linear Schrödinger equation. Looking at the equations, however, they rather appear to describe a non-linear interaction of waves combining aspects of the Korteweg-de Vries and the modified Korteweg-de Vries equations. $a = \pm b$ gives the modified Korteweg-de Vries equation, while b = constant leaves us with the Korteweg-de Vries equation [6, 13].

To illustrate another point, let us consider the system obtained by keeping

(F) $\qquad \xi^4, \xi^5, \xi^7 \sim \xi^9 \qquad \xi^{12} = \xi^8 \neq 0.$

After solving (3.2b) and finding exact 1-forms to be used as coordinate differentials we obtain

$$\xi^4 = \mathrm{d}y = a\,\mathrm{d}x + (c - \tfrac{1}{4}a^3)\,\mathrm{d}t$$

$$\xi^5 = \mathrm{d}x - \tfrac{1}{2}(b + \tfrac{1}{2}a^2)\,\mathrm{d}t$$

$$\xi^7 = a\,\mathrm{d}t$$

$$\xi^8 = \mathrm{d}t$$

$$\xi^9 = \mathrm{d}x + \tfrac{1}{2}(b - \tfrac{1}{2}a^2)\,\mathrm{d}t.$$

We have three exact 1-forms, $\mathrm{d}y$, $\mathrm{d}x$ and $\mathrm{d}t$, two of which are independent. It is now our choice which two to use as coordinates; the third one becomes a potential. Using x and t, as suggested by the way the ξ are written above, gives the modified Korteweg-de Vries equation

$$a_t = \tfrac{1}{2}(a_{xxx} - \tfrac{3}{2}a^2 a_x). \tag{3.5}$$

This is not be confused with case (C). The equation obtained there is the one for $u = \int a\,\mathrm{d}x$.

If we choose y instead of x as coordinate we have

$$\xi^4 = \mathrm{d}y \qquad \xi^5 = (1/a)\,\mathrm{d}y - (c/a + \tfrac{1}{2}b)\,\mathrm{d}t \qquad \xi^7 = a\,\mathrm{d}t$$

$$\xi^8 = \mathrm{d}t \qquad \xi^9 = (1/a)\,\mathrm{d}y - (c/a - \tfrac{1}{2}b)\,\mathrm{d}t.$$

The differential system then yields

$$a_y = (1/a)b$$

$$(1/a)a_t - (c/a)_y - \tfrac{1}{2}b_y = -c/a - \tfrac{1}{2}b$$

$$(1/a^2)a_t - (c/a)_y + \tfrac{1}{2}b_y = c/a - \tfrac{1}{2}b.$$

It is now advantageous to introduce $u = \sqrt{a}$ which gives

$$u_t = \tfrac{1}{2}u^{3/2}(u_{yyy} - u_y). \tag{3.6}$$

This transformation interchanging a coordinate, x, with a potential, y, is analogous to the one connecting the modified Korteweg-de Vries equation of case (C) with the Harry-Dym equation [9] or the one mentioned in [2]. Transformations of this type are rather difficult to find if one works with the equations directly [10]. On the other hand, they are almost obvious from the differential system.

To demonstrate another possibility within the present formalism we take

(G) $\xi^4, \xi^5, \xi^7 \sim \xi^{10} \qquad \xi^{12} \neq 0$

for which the ξ can be calculated as above and are given by

$$\xi^4 = dz' \qquad \xi^5 = a\,dx + (e - a^2 b)\,dt \qquad \xi^7 = dx - ab\,dt$$

$$\xi^8 = a\,dt \qquad \xi^9 = b\,dx + (f - ab^2)\,dt \qquad \xi^{10} = dt \qquad \xi^{12} = b\,dt.$$

Moreover it can be shown that

$$d[ab\xi^7 + (af + be - \tfrac{1}{2}a^2b^2)\xi^{10}] = 0$$

or

$$dw = ab\,dx + (af + be - \tfrac{3}{2}a^2b^2)\,dt.$$

Rather than using w to replace x as coordinate, as has been done above, we take $z = z' - \alpha w$ with constant α as coordinate. This gives

$$\xi^4 = dz + \alpha ab\,dx + \alpha(af + be - \tfrac{3}{2}a^2b^2).$$

The equations to be derived from (3.2a) are

$$e = a_x - \alpha a^2 b \qquad a_z = a \qquad e_z = e$$

$$f = -b_x - \alpha ab^2 \qquad b_z = -b \qquad f_z = -f$$

$$a_t = e_x - 2aba_x - a^2b_x + \alpha a^2 f - \tfrac{1}{2}\alpha a^3 b^2$$

$$b_t = f_x - 2abb_x - b^2a_x - \alpha b^2 e + \tfrac{1}{2}\alpha a^2 b^3.$$

Finally they are

$$\begin{aligned} a_t &= a_{xx} - 2ab(\alpha+1)a_x - (2\alpha+1)a^2b_x - \alpha(\alpha+\tfrac{1}{2})a^3b^2 \\ b_t &= -b_{xx} - 2ab(\alpha+1)b_x - (2\alpha+1)b^2a_x + \alpha(\alpha+\tfrac{1}{2})a^2b^3. \end{aligned} \tag{3.7}$$

There are three more or less obvious choices for the parameter α, namely

$$\alpha = 0 \qquad \begin{cases} a_t = (a_x - a^2b)_x \\ b_t = -(b_x + ab^2)_x \end{cases} \tag{3.8a}$$

$$\alpha = -\tfrac{1}{2} \qquad \begin{cases} a_t = a_{xx} - aba_x \\ b_t = -b_{xx} - abb_x \end{cases} \tag{3.8b}$$

$$\alpha = -1 \qquad \begin{cases} a_t = a_{xx} + a^2b_x - \tfrac{1}{2}a^3b^2 \\ b_t = -b_{xx} + b^2a_x + \tfrac{1}{2}a^2b^3. \end{cases} \tag{3.8c}$$

The last equation is known as the derivative non-linear Schrödinger equation and can be solved by an inverse scattering transformation [11]. Note that two sets of (3.7) with different α are connected by a transformation of the type

$$\tilde{a} = a\, e^{\text{constant} \times w} \qquad \tilde{b} = b\, e^{-\text{constant} \times w}.$$

A similar transformation is also possible for the non-linear Schrödinger equation.

As an example of a system for which reduction is essential we mention

(H) $\xi^3 \sim \xi^6 \qquad \xi^8 \neq 0.$

The genus of this system is three. We can, however, find only two exact 1-forms. ξ^4 is independent of the other forms and the right-hand side of $d\xi^4$ is independent of the other right-hand sides in (3.2a). Hence we reduce the system by setting

$$\zeta = \alpha\xi^3 + \xi^4 + \gamma\xi^5 + \delta\xi^6 + \varepsilon\xi^8.$$

The coefficient of ξ^4 has been scaled to unity. The condition

$$\zeta\, d\zeta = 0 \bmod(\Omega, \Sigma)$$

gives

$$2\gamma\delta = 2\alpha\varepsilon = -1.$$

If we now put $\zeta = 0$ and find coordinates for the other forms we obtain

$$\xi^3 = e^{\phi}\, dx \qquad \xi^6 = e^{\psi}\, dt \qquad \xi^5 = e^{-\phi}\, dx \qquad \xi^8 = e^{-\psi}\, dt$$

$$\xi^4 = (1/\sqrt{2})[(l\, e^{\phi} - k\, e^{-\phi})\, dx + (e^{\psi}/k - e^{-\psi}/l)\, dt].$$

The equations implied by (3.2a) are

$$\phi_t = -(1/\sqrt{2})(e^{\psi}/k - e^{-\psi}/l)$$

$$\psi_x = -(1/\sqrt{2})[l\, e^{\phi} - k\, e^{-\phi}].$$

This can be recognised as the standard form a Bäcklund transformation—not in the sense of § 1—between two sinh-Gordon equations because it follows that

$$(\phi + \psi)_{xt} = (l/k)\, e^{\phi+\psi} - (k/l)\, e^{-(\phi+\psi)}$$

$$(\phi - \psi)_{xt} = e^{\phi-\psi} - e^{-(\phi-\psi)}.$$

Without going into the details we just note that the following equations can be derived.

(I) $\xi^3 \sim \xi^8 \neq 0$

$$a_{xt} = aba_x \qquad b_{xt} = -abb_x.$$

(J) $\xi^1, \xi^3 \sim \xi^8 \neq 0$

$$a_{xt} = a + aba_x \qquad b_{xt} = b - abb_x.$$

(K) $\xi^3 \sim \xi^8, \xi^3 = \xi^5 \neq 0$

$$aa_{xxt} - a_x(a_{xt} + 2) = a^3 a_t.$$

(L) $\xi^4, \xi^6, \xi^7, \xi^9 = \xi^{10} \neq 0$

$$a_t + a_{xx} - 2aa_x = 0$$

or, interchanging potential and coordinate,

$$a_t = a^2(a_{zz} - a_z).$$

This list is, of course, by no means exhaustive, it merely serves as an indication of what can be done.

4. A Bäcklund transformation

As an example of how the present formalism works for deriving Bäcklund transformations we shall consider case (E) from the previous section. With the explicit 1-forms given in (3.3) we find for the pseudopotentials

$$\mathrm{d}y = (Y_0 a + Z_0 b + X_1)\,\mathrm{d}x + [X_3 - (af+be)X_0 - abX_1 + (g-a^2b)Y_0 + eY_1 + aY_2 + (h-ab^2)Z_0 + fZ_1 + bZ_2]\,\mathrm{d}t. \tag{4.1}$$

The 2-forms F relevant for the system (3.4) are

$$\begin{aligned} &-\mathrm{d}a\,\mathrm{d}t + e\,\mathrm{d}x\,\mathrm{d}t \qquad -\mathrm{d}b\,\mathrm{d}t - f\,\mathrm{d}x\,\mathrm{d}t \qquad -\mathrm{d}e\,\mathrm{d}t + g\,\mathrm{d}x\,\mathrm{d}t\\ &-\mathrm{d}f\,\mathrm{d}t - h\,\mathrm{d}x\,\mathrm{d}t \qquad -\mathrm{d}a\,\mathrm{d}x - \mathrm{d}g\,\mathrm{d}t + 3abe\,\mathrm{d}x\,\mathrm{d}t\\ &-\mathrm{d}b\,\mathrm{d}x - \mathrm{d}h\,\mathrm{d}t - 3abf\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.2}$$

With the map $\phi : (a, b, e, f, g, h, y) \to (\tilde{a}, \tilde{b}, \tilde{e}, \tilde{f}, \tilde{g}, \tilde{h})$, the condition (1.8) for this to constitute a Bäcklund transformation becomes

$$\tilde{e} = \tilde{a}_a e - \tilde{a}_b f + \tilde{a}_{Y_0} a + \tilde{a}_{Z_0} b + \tilde{a}_{X_1} \tag{4.3a}$$

$$\tilde{f} = -\tilde{b}_a e + \tilde{b}_b f - \tilde{b}_{Y_0} a - \tilde{b}_{Z_0} b - \tilde{b}_{X_1} \tag{4.3b}$$

$$\tilde{g} = \tilde{e}_a e - \tilde{e}_b f + \tilde{e}_e g - \tilde{e}_f h + \tilde{e}_{Y_0} a + \tilde{e}_{Z_0} b + \tilde{e}_{X_1} \tag{4.3c}$$

$$\tilde{h} = -\tilde{f}_a e + \tilde{f}_b f - \tilde{f}_e g + \tilde{f}_f h - \tilde{f}_{Y_0} a - \tilde{f}_{Z_0} b - \tilde{f}_{X_1} \tag{4.3d}$$

$$\begin{aligned} &3\tilde{a}_a abe - 3\tilde{a}_b abf - \tilde{a}_{X_3} + \tilde{a}_{X_0}(af+be) + \tilde{a}_{X_1} ab\\ &\qquad - \tilde{a}_{Y_0}(g-a^2b) - \tilde{a}_{Y_1} e - \tilde{a}_{Y_2} a - \tilde{a}_{Z_0}(h-ab^2) - \tilde{a}_{Z_1} f - \tilde{a}_{Z_2} b\\ &\quad = -\tilde{g}_a e + \tilde{g}_b f - \tilde{g}_e g + \tilde{g}_f h - \tilde{g}_{Y_0} a - \tilde{g}_{Z_0} b - \tilde{g}_{X_1} + 3\tilde{a}\tilde{b}\tilde{e} \end{aligned} \tag{4.3e}$$

$$\begin{aligned} &3\tilde{b}_a abe - 3\tilde{b}_b abf - \tilde{b}_{X_3} + \tilde{b}_{X_0}(af+be) + \tilde{b}_{X_1} ab\\ &\qquad - \tilde{b}_{Y_0}(g-a^2b) - \tilde{b}_{Y_1} e - \tilde{b}_{Y_2} a - \tilde{b}_{Z_0}(h-ab^2) - \tilde{b}_{Z_1} f - \tilde{b}_{Z_2} b\\ &\quad = -\tilde{h}_a e + \tilde{h}_b f - \tilde{h}_e g + \tilde{h}_f h - \tilde{h}_{Y_0} a - \tilde{h}_{Z_0} b - \tilde{h}_{X_1} - 3\tilde{a}\tilde{b}\tilde{f} \end{aligned} \tag{4.3f}$$

$$\begin{aligned} &\tilde{a}_e = \tilde{a}_f = \tilde{a}_g = \tilde{a}_h = \tilde{b}_e = \tilde{b}_f = \tilde{b}_g = \tilde{b}_h = 0\\ &\tilde{e}_g = \tilde{e}_h = \tilde{f}_g = \tilde{f}_h = 0\\ &\tilde{a}_a = \tilde{g}_g \qquad \tilde{a}_b = \tilde{g}_h\\ &\tilde{b}_a = \tilde{h}_g \qquad \tilde{b}_b = \tilde{h}_n. \end{aligned} \tag{4.3g}$$

Here a_{Gi}, etc, denotes the directional derivative $a_{y_\alpha} G_i^\alpha$, etc. Differentiating (4.3e) with respect to g and then with respect to e or f we obtain the expressions

$$\tilde{g}_{ee} = -\tilde{a}_{aa} \qquad \tilde{g}_{ef} = \tilde{a}_{ab} \qquad \tilde{g}_{ff} = -\tilde{a}_{bb}.$$

Substituting (4.3c) gives the result that $\tilde{a}$ has to be a linear function of a and b. Moreover we find that either the coefficient of a or that of b has to vanish. Hence we set

$$\tilde{a} = \alpha(y)a + k(y)$$

but then we get again from (4.3e) differentiated with respect to g that

$$\alpha_{Y_0} = \alpha_{Z_0} = \alpha_{X_1} = 0.$$

As $\alpha_{X_0}=\alpha_{Y_0Z_0}-\alpha_{Z_0Y_0}=0$, $\alpha_{Z_1}=\alpha_{Z_0X_1}-\alpha_{X_1Z_0}=0$, etc, we conclude that α is constant. Without loss of generality (an analogous result holds for b) we obtain

$$\tilde{a}=a+k(y) \qquad \tilde{b}=b+l(y).$$

Inserting this into (4.3) we obtain among other equations

$$\begin{aligned} &l_{Y_0}=0 && l_{Z_0Y_0}=l=-l_{X_0} && l_{Z_0X_1}=kl \\ &k_{Z_0}=0 && k_{Y_0Z_0}=k=k_{X_0} && k_{Y_0X_1}=kl \\ &k_{Y_0}=l_{Z_0}. \end{aligned}$$

It is now clear that the reduction to a three-dimensional algebra (2.7) does not yield a non-trivial solution, e.g.

$$l_{Z_0X_1}=\alpha l_{Z_0X_0}=\alpha(l_{X_0Z_0}+l_{Z_0})=\alpha(-l_{Z_0}+l_{Z_0})=0.$$

Hence we have to use (2.8). The easiest case is to take $\nu+\mu^2=\lambda^2>0$ which gives $O(2,1)\otimes O(2,1)$ as the six-dimensional algebra. Using a one-dimensional non-linear representation for the O(2, 1), we find the vectors with coordinates u and v

$$\begin{aligned} &X_0=\begin{bmatrix}1\\-1\end{bmatrix} && Y_0=-\frac{1}{\sqrt{2}}\begin{bmatrix}e^{-u}\\e^{v}\end{bmatrix} && Z_0=\frac{1}{\sqrt{2}}\begin{bmatrix}e^{u}\\e^{-v}\end{bmatrix} \\ &X_1=\begin{bmatrix}\lambda+\mu\\\lambda-\mu\end{bmatrix}. && Y_1=-\frac{1}{\sqrt{2}}\begin{bmatrix}(\mu+\lambda)\,e^{-u}\\(\mu-\lambda)\,e^{v}\end{bmatrix} && Z_1=\frac{1}{\sqrt{2}}\begin{bmatrix}(\mu+\lambda)\,e^{u}\\(\mu-\lambda)\,e^{-v}\end{bmatrix}. \end{aligned} \tag{4.4}$$

Without going into the detailed calculations we note that the remaining equations require $\mu=0$ and that the Bäcklund transformation is given by

$$\tilde{a}=a-\frac{2\sqrt{2}\lambda}{e^{v}+e^{-u}} \qquad \tilde{b}=b-\frac{2\sqrt{2}\lambda}{e^{u}+e^{-v}}. \tag{4.5}$$

Substituting the vectors (4.4) into (4.1) gives

$$\begin{aligned} du=&\left(-\frac{1}{\sqrt{2}}e^{-u}a+\frac{1}{\sqrt{2}}e^{u}b+\lambda\right)dx+\bigg(\lambda^3+(ab_x-ba_x)-\lambda ab \\ &-\frac{1}{\sqrt{2}}(a_{xx}-a^2b+\lambda a_x+\lambda^2a)\,e^{-u}+\frac{1}{\sqrt{2}}(b_{xx}-ab^2-\lambda b_x+\lambda^2b)\,e^{u}\bigg)\,dt \end{aligned} \tag{4.6a}$$

$$\begin{aligned} dv=&\left(-\frac{1}{\sqrt{2}}e^{v}a+\frac{1}{\sqrt{2}}e^{-v}b+\lambda\right)dx+\bigg(\lambda^3-(ab_x-ba_x)-\lambda ab \\ &-\frac{1}{\sqrt{2}}(a_{xx}-a^2b-\lambda a_x+\lambda^2a)\,e^{v}+\frac{1}{\sqrt{2}}(b_{xx}-ab^2+\lambda b_x+\lambda^2b)\,e^{-v}\bigg)\,dt. \end{aligned} \tag{4.6b}$$

Equations (4.6a) and (4.6b) can be converted into Riccati equations for e^u and e^v respectively. Linear equations can be obtained by using linear representations for the algebras.

Let us take as a simple seed solution

$$a=-\sqrt{2}a_0=\text{constant} \qquad b=0.$$

Then we obtain the new solutions

$$\tilde{a} = -\sqrt{2}\,\frac{a_0 \cosh(\lambda x + \lambda^3 t) - 1}{\cosh(\lambda x + \lambda^3 t) + a_0}$$

$$\tilde{b} = \sqrt{2}\,\frac{\lambda^2}{\cosh(\lambda x + \lambda^3 t) + a_0}.$$

5. Concluding remarks

We shall now briefly explain, how the formalism of Ablowitz *et al* [6, 7, 12, 13] is related to the present one. They start with two linear equations (the notation of [7] has been changed)

$$r_x = Ur \qquad r_t = Vr \tag{5.1}$$

for an n-dimensional vector r and $n \times n$ matrices U, V. Cross differentiation yields

$$U_t - V_x + [UV] = 0. \tag{5.2}$$

For a 2×2 eigenvalue problem with constant eigenvalue λ they consider

$$U = \begin{bmatrix} -\mathrm{i}\lambda & q \\ p & \mathrm{i}\lambda \end{bmatrix} \qquad V = \begin{bmatrix} E & F \\ G & -E \end{bmatrix}$$

or

$$U = \begin{bmatrix} -\mathrm{i}\lambda & q\lambda \\ p\lambda & \mathrm{i}\lambda \end{bmatrix} \qquad V = \begin{bmatrix} E & F \\ G & -E \end{bmatrix}.$$

They then expand V in a truncated power series of λ and λ^{-1} and derive non-linear equations for p and q from the compatibility condition (5.2) by setting the coefficients of the various powers of λ to zero. The inverse scattering transform can then be applied.

Matrix equations like (5.1) are reminiscent of (1.3) and (1.15) and arise naturally in the present context. In fact they can be constructed from the infinite algebra and the Ω, Σ system. A representation of our algebra in terms of infinite matrices whose elements are 2×2 matrices is easy to find (cf [1])

$$X_i = (k\delta^{\beta}_{\alpha-i}) \qquad Y_i = (l\delta^{\beta}_{\alpha-i}) \qquad Z_i = (m\delta^{\beta}_{\alpha-i}) \qquad -\infty < \alpha, \beta < \infty$$

where the 2×2 matrices k, l, m satisfy

$$[k, l] = l \qquad [k, m] = -m \qquad [l, m] = k.$$

This yields immediately a representation in terms of infinitely many two-component vectors r^α

$$X_i = (kr^{\alpha-i}) \qquad Y_i = (lr^{\alpha-i}) \qquad Z_i = (mr^{\alpha-i}). \tag{5.3}$$

The r^α are indeed the prolongation variables y from § 1.

In the notation of § 1 equation (1.15) is $\mathrm{d}y = X_i\xi^i$. There is no infinite sum over i, as for a given equation only a finite number of ξ do not vanish. For instance, case (F) of § 3 gives

$$\mathrm{d}y = X_0\xi^4 + Y_0\xi^5 + X_1\xi^7 + (Y_1 + Z_2)\xi^8 + Z_1\xi^9$$

or, using x and t as coordinates,

$$\mathrm{d}r^{\alpha} = [(ak+l)r^{\alpha} + mr^{\alpha-1}]\,\mathrm{d}x + \{[(c-\tfrac{1}{4}a^3)k - \tfrac{1}{2}(b+\tfrac{1}{2}a^2)l]r^{\alpha} + mr^{\alpha-2} + [ak+l+\tfrac{1}{2}(b-\tfrac{1}{2}a^2)m]r^{\alpha-1}\}\,\mathrm{d}t.$$

Assuming that $r(\lambda) = \Sigma_{\alpha=-\infty}^{\alpha=\infty} r^{\alpha}\lambda^{\alpha}$ converges for some $\lambda_1 < \lambda < \lambda_2$, we can write

$$\mathrm{d}r(\lambda) = (ak+l+\lambda m)r(\lambda)\,\mathrm{d}x + \{(c-\tfrac{1}{4}a^3+\lambda a)k + [\lambda - \tfrac{1}{2}(b+\tfrac{1}{2}a^2)]l + \lambda[\lambda + \tfrac{1}{2}(b-\tfrac{1}{2}a^2)]m\}r(\lambda)\,\mathrm{d}t. \tag{5.4a}$$

On the other hand, for y and t as coordinates we obtain

$$\mathrm{d}r(\lambda) = [k+(1/a)l+\lambda/a - m]r(\lambda)\,\mathrm{d}y + \{a\lambda k + [\lambda - (c/a+\tfrac{1}{2}b)]l + \lambda[\lambda - (c/a-\tfrac{1}{2}b)]m\}r(\lambda)\,\mathrm{d}t. \tag{5.4b}$$

Both expressions (5.4) have the structure of a Lax pair. How one is connected to the other is by no means obvious unless one uses the Ω, Σ system.

The relation between the approach of Ablowitz *et al* and the present one can now be explained as follows: Ablowitz *et al* start with (1.3) (where (5.1) is the same equation written in matrix form) and assume that A and B can be expanded in a truncated power series of an eigenvalue parameter. Equation (1.5) then yields the 2-forms which when annulled give the desired equations. The approach via prolongation structures truncates the Cartan-Maurer forms of an infinite-dimensional algebra and reconstructs the equations from the closed ideal of the remaining forms, the Ω, Σ system.

If it were only for the linear equation (5.1) the two methods would be equivalent. However, the Estabrook-Wahlquist approach has some advantages. For instance, transformations of the type 'interchange of coordinate and potential' are almost obvious and transformations of the Miura type are easily recognised (cf [2]); it is also easy to see that equations (3.8) stem from the same system.

It should be noted that, if forms dual to generators with negative index are retained, the resulting equations are of the wave equation type. In the other case they are of the evolution equation type.

It is clear that many more equations can be derived by the present method. With the recent interest in non-linear differential equations it appears useful to have methods for deriving equations which admit an infinite number of conserved quantities, Bäcklund transformations are solvable by the inverse scattering method. Other infinite-dimensional algebras may, of course, also be taken as prolongation structures and more equations, some of them perhaps interesting or useful, can be constructed.

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